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Difference Posets and the Histories Approach to Quantum Theories

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Direct limits and tensor products of difference posets are studied. In the spirit of a recent paper by Isham, a potential model for an "unsharp histories" approach to quantum theory based on difference posets as abstract models for the set of effects is considered. It is shown that the set of all histories in this approach has an algebraic structure of a difference poset.

1. INTRODUCTION

In the historic paper by Birkhoff and von Neumann (1936), the notion of *quantum logic* was introduced to the description of quantum mechanical events. In the axiomatic approach to quantum mechanics, the event structure of a physical system is identified with a quantum logic [a σ-orthomodular poset or lattice (Pták and Pulmannová, 1991; Varadarajan, 1968/1970)]. More general structures, *orthoalgebras*, have been introduced by Foulis and Randall (1972; Randall and Foulis, 1973) and they enable one to introduce a tensor product (Bennett and Foulis, 1993), which is an important tool to describe coupled systems.

Events of quantum logics or orthoalgebras have a "yes–no" character and therefore they do not describe unsharp measurements. To include them, the set of all *effects* is to be considered in the Hilbert space approach to quantum mechanics (Busch *et al.*, 1991), i.e., the set $\mathscr{E}(H)$ of all self-adjoint operators on the Hilbert space H with spectra in the interval $[0, 1]$. Then "yes–no" events, i.e., those having spectrum in the two-point set $\{0, 1\}$, correspond to orthogonal projection operators on H.

Recently, there has appeared a new mathematical model, *difference posets* (or D-posets, for short), introduced in Kôpka and Chovanec (1994).

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D-posets generalize quantum logics and orthoalgebras as well as the set of all effects. In this model, the difference operation is a primary notion from which we derive other, usual notions important for measurements. We note that the same structure, called "effect algebra," can be obtained on the basis of another partial binary operation, a "plus" operation (Foulis and Bennett, 1994; Giuntini and Greuling, 1989; Pulmannová, 1994; Hedlíková and Pulmannová, n.d.), which appears also in orthoalgebras.

In a recent paper by Isham (n.d.) it is shown that the Gell-Mann and Hartle (1990*a*–*c*) axioms for a generalized "histories" approach to quantum theory can be modified in such a way that each history proposition in the standard approach is represented by a genuine projection operator. This provides a valuable insight into the algebraic structure of general history theories, and also provides a number of potential models for theories of this type. Our aim is to present one of those models in the present paper. We use the partial algebraic structure of D-posets as models for the sets of all quantum mechanical effects. A *homogeneous history* (or a *history filter*) is modeled by a finite sequence of elements of a D-poset. We introduce a modification of Isham's axioms. Under these axioms, the set of all histories admits a structure of a D-poset. We show that it can be obtained by a construction of a direct limit of a directed system of finite tensor products of D-posets. The standard approach is obtained as a special case.

We note that since effects represent unsharp measurements, we obtain "unsharp" histories, where a history can "exclude" itself. At this point, a many-valued logic comes into the picture. However, we provide a purely algebraic description; logical and philosophical analysis is not the subject of the present paper.

2. BASIC FACTS ABOUT D-POSETS

A *D-poset*, or a *difference poset*, is a partially ordered set[2] L with a partial ordering $\leq$, the greatest element 1, and with a partial binary operation $\ominus: L \times L \to L$, called a *difference*, such that, for $a, b \in L$, $b \ominus a$ is defined if and only if $a \leq b$, and the following axioms hold for $a, b, c \in L$:

(DPi) $b \ominus a \leq b$.

(DPii) $b \ominus (b \ominus a) = a$.

(DPiii) $a \leq b \leq c \Rightarrow c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

The following statements have been proved in Kôpka and Chovanec (1994).

[2] As usual, we shall assume that card $L \geq 2$.

Proposition 2.1. Let a, b, c, d be elements of a D-poset L. Then:

(i) $1 \ominus 1$ is the smallest element of L; denote it by 0.
(ii) $a \ominus 0 = a$.
(iii) $a \ominus a = 0$.
(iv) $a \leq b \Rightarrow b \ominus a = 0 \Leftrightarrow b = a$.
(v) $a \leq b \Rightarrow b \ominus a = b \Leftrightarrow a = 0$.
(vi) $a \leq b \leq c \Rightarrow b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.
(vii) $b \leq c$, $a \leq c \ominus b \Rightarrow b \leq c \ominus a$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.
(viii) $a \leq b \leq c \Rightarrow a \leq c \ominus (b \ominus a)$ and $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.

Remark 2.2 (Navara and Ptàk, n.d.). A poset L with the smallest and greatest elements 0 and 1, respectively, and with a partial binary operation $\ominus$: $L \times L \to L$ such that $b \ominus a$ is defined iff $a \leq b$, and for $a, b, c \in L$ we have

(i) $a \ominus 0 = a$,
(ii) if $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$,

is a D-poset.

For any element $a \in L$ we put

$$a^{\perp} := 1 \ominus a$$

Then (i) $a^{\perp\perp} = a$; (ii) $a \leq b$ implies $b^{\perp} \leq a^{\perp}$. Two elements a and b of L are *orthogonal*, and we write $a \perp b$, iff $a \leq b^{\perp}$ (iff $b \leq a^{\perp}$).

Now we introduce a partial binary operation $\oplus$: $L \times L \to L$ such that an element $c = a \oplus b$ in L is defined iff $a \perp b$, and for c we have $b \leq c$ and $a = c \ominus b$. The partial operation $\oplus$ is defined correctly because if there exists $c_1 \in L$ with $b \leq c_1$ and $a = c_1 \ominus b$, then, by (viii) of Proposition 2.1 and (DPii), we have

$$(1 \ominus (c \ominus b)) \ominus b = 1 \ominus c = (1 \ominus (c_1 \ominus b)) \ominus b = 1 \ominus c_1$$

which implies $c = c_1$. Moreover,

$$c = a \oplus b = (a^{\perp} \ominus b)^{\perp} = (b^{\perp} \ominus a)^{\perp} \tag{2.1}$$

The operation $\oplus$ is commutative and associative. Very important examples of difference posets are orthomodular posets (= quantum logics), orthoalgebras, and sets of effects.

Example 2.3. An *orthomodular poset* (OMP), that is, a partially ordered set L with an ordering $\leq$, the smallest and greatest elements 0 and 1, respectively, and an orthocomplementation $\perp: L \to L$ such that

(OMi) $a^{\perp\perp} = a$ for any $a \in L$,
(OMii) $a \vee a^{\perp} = 1$ for any $a \in L$,
(OMiii) if $a \leq b$, then $b^{\perp} \leq a^{\perp}$,
(OMiv) if $a \leq b^{\perp}$ (and we write $a \perp b$), then $a \vee b \in L$,
(OMv) if $a \leq b$, then $b = a \vee (a \vee b^{\perp})^{\perp}$ (orthomodular law),

is a D-poset, when $b \ominus a := b \wedge a^{\perp}$.

Example 2.4. An *orthoalgebra*, that is, a set L with two particular elements 0, 1, and with a partial binary operation $\oplus: L \times L \to L$ such that for all $a, b, c \in L$ we have

(OAi) if $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (commutativity),
(OAii) if $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity),
(OAiii) for any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation),
(OAiv) if $a \oplus a$ is defined, then $a = 0$ (consistency),

is a D-poset if $b \ominus a := (a \oplus b^{\perp})^{\perp}$, where $b^{\perp}$ is a unique element c in L such that $b \oplus c = 1$.

If the assumptions of (OAii) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L.

We note that if L is an orthomodular poset and $a \oplus b := a \vee b$ whenever $a \perp b$ in L, then L with 0, 1, $\oplus$ is an orthoalgebra. The converse statement does not hold, in general. We recall that an orthoalgebra L is an OMP iff $a \perp b$ implies $a \vee b \in L$.

By Navara and Pták (n.d.) we conclude that a D-poset L with 0, 1, and $\oplus$, defined by (2.1), is an orthoalgebra if and only if $a \leq 1 \ominus a$ implies $a = 0$. Therefore, it is not hard to give many examples of D-posets which are not orthoalgebras; such ones are sets of effects:

Example 2.5. The set $\mathscr{E}(H)$ of all Hermitian operators A on H such that $O \leq A \leq I$, where I is the identity operator on H, is a difference poset which is not an orthoalgebra; a partial ordering $\leq$ is defined via $A \leq B$ iff $(Ax, x) \leq (Bx, x)$, $x \in H$, and $C = B \ominus A$ iff $(Ax, x) - (Bx, x) = (Cx, x)$, $x \in H$.

On the other hand, if in the definition of an orthoalgebra, axiom (OAiv) is replaced by a weaker axiom

(EAiv) $a \oplus 1$ is defined implies $a = 0$

we obtain so-called *effect algebra* or *weak orthoalgebra* (Foulis and Bennett, 1994; Giuntini and Greuling, 1989), which is equivalent to a D-poset (Foulis and Bennett, 1994; Pulmannová, 1994).

Let A, B be D-posets. A mapping $f: A \to B$ is a *morphism* if

$$\exists b \ominus a \Rightarrow \exists f(b) \ominus f(a) \quad \text{and} \quad f(b \ominus a) = f(b) \ominus f(a)$$

A morphism $f: A \to B$ is a *full morphism* if

$$\exists f(b) \ominus f(a) \quad \text{and} \quad f(b) \ominus f(a) \in f[A]$$
$$\Rightarrow \exists a_1, b_1 \in A \quad \text{such that} \quad \exists b_1 \ominus a_1 \quad \text{and} \quad f(a) = f(a_1), f(b) = f(b_1)$$

A morphism $f: A \to B$ is a *closed morphism* (or a *monomorphism*) if

$$\exists f(b) \ominus f(a) \Rightarrow \exists b \ominus a$$

It is easily seen that if $f: A \to B$ is a morphism, then:

(i) $f(1) = 1$.
(ii) $f(a^\perp) = f(a)^\perp$.
(iii) $\exists a \oplus b$ implies $\exists f(a) \oplus f(b)$ and $f(a \oplus b) = f(a) \oplus f(b)$.

We recall that a morphism f is an *isomorphism* if it is a bijection and f^{-1} is also a morphism. Equivalently, if f is surjective and closed.

Let A be a D-poset. A relation $R \subset A \times A$ is called a *congruence* (*relation*) on A if it satisfies the following:

(i) R is an equivalence relation on A.
(ii) If a_1Rb_1, a_2Rb_2 and $a_2 \ominus a_1$, $b_2 \ominus b_1$ exist, then $a_2 \ominus a_1Rb_2 \ominus b_1$.

R will be called a *closed congruence* iff R satisfies in addition the following:

(iii) If a_1Rb_1, a_2Rb_2 and $a_2 \ominus a_1$ exists, then $b_2 \ominus b_1$ exists.

We note that (iii) is equivalent to the following:

(iii)* a_1Rb_1, $a_1 \perp a_2 \Rightarrow b_1 \perp a_2$.

Let A be a D-poset and R a congruence on A. By $\tilde{a}$ we denote the congruence class of $a \in A$ with respect to R, and we define as usual $A/R := \{\tilde{a}: a \in A\}$, the set of all congruence classes of elements of A, and $h: A \to A/R$, $a \to \tilde{a}$ the *natural projection* from A onto A/R. In order to get a $\ominus$-operation on A/R, the *quotient* of A with respect to R, we define

$\tilde{b} \ominus \tilde{a}$ is defined iff there are $b_1 \in \tilde{b}$, $a_1 \in \tilde{a}$ such that $b_1 \ominus a_1$ is defined, and then $\tilde{b} \ominus \tilde{a} = (b_1 \ominus a_1)^\sim = h(b_1 \ominus a_1)$

Then $\ominus$ is well-defined and the natural projection h: $A \to a/R$ is a full and surjective morphism. h is a closed morphism iff R is a closed congruence.[3]

For more details about morphisms and congruences of partial algebras see Burmester (1986).

Definition 2.6. (i) A *directed system* (of D-posets) is a family $A_{\underline{I}} :=$ $(A_i, f_{ij}: A_i \to A_j,\ i, j \in I,\ i \leq j)$, where $\underline{I} := (I, \leq)$ is a directed poset, A_i is a D-poset for each $i \in I$, and each f_{ij} is a morphism $(i \leq j)$, such that:

(i1) $f_{ii} = 1_{A_i}$ for every $i \in I$.
(i2) If $i \leq j \leq m$ in I, then $f_{jm} f_{ij} = f_{im}$.

(ii) Let $A_{\underline{I}}$ be a directed system of D-posets; then $\underline{f} := ((f_i: A_i \to A;\ i \in I), A)$ is called its *direct limit* iff the following hold:

(ii1) A is a D-poset; f_i is a morphism for each i $\in I$.
(ii2) If $i \leq j$ in I, then $f_j f_{ij} = f_i$ (i.e., $\underline{f}$ is *compatible* with $A_{\underline{I}}$).
(ii3) *If* $\underline{g} := ((g: A_i \to B,\ i \in I), B)$ is any system compatible with $A_{\underline{I}}$ (i.e., $g_j f_{ij} = g_i$ for all $i \leq j$ in I), then there exists exactly one morphism g: $A \to B$ such that $g f_i = g_i$ for every $i \in I$ (i.e., one has commutativity of the diagram in Fig. 1).

Often only the object A above is denoted by $\lim_{\to} A_{\underline{I}}$.

It is easy to see that every direct limit is unique up to isomorphism. The existence of direct limits can be obtained from more general consideration (e.g., Burmester, 1986). For the convenience of the reader, we give here a proof specialized to D-posets.

Theorem 2.7. Let $A_{\underline{I}}$ be a directed system of D-posets, where f_{ij} is a morphism for every $i, j \in I$, $i \leq j$. Then a direct limit exists.

Proof. Put $A = \bigcup_{i \in I} A_i$ and define a relation $\equiv$ on A as follows: we put $a \equiv b$ $(a \in A_i,\ b \in A_j)$ if there is $k \in I$ with $i, j \leq k$ such that $f_{ik}(a) = f_{jk}(b)$

$$\begin{tikzcd}
A_i \arrow[r, "g_i"] \arrow[dr, "f_i"'] & B \\
& A \arrow[u, "g"']
\end{tikzcd}$$

Fig. 1

[3]We note that to get a D-poset structure on A/R, some more conditions on R are needed, in general (see Pulmannová, n.d.).

in A_k. Let us choose an arbitrary $k_1 \in I$ with $i, j \leq k_1$. Then for any $l \in I$ with $k, k_1 \leq l$, the following equalities hold:

$$f_{il}(a) = f_{k_1 l} f_{ik_1}(a)$$

$$f_{il}(a) = f_{kl} f_{ik}(a) = f_{kl} f_{jk}(b) = f_{jl}(b)$$

$$f_{jl}(b) = f_{k_1 l} f_{jk_1}(b)$$

Hence

$$f_{ik_1}(a) \equiv f_{jk_1}(b)$$

Reflexivity and symmetry of the relation $\equiv$ are evident. To prove transitivity, let $a \equiv b$ and $b \equiv c$, where $a \in A_i$, $b \in A_j$, $c \in A_k$. Then there are $l_1, l_2 \in I$ with $i, j \leq l_1, j, k \leq l_2$ and $f_{il_1}(a) = f_{jl_1}(b)$ and $f_{jl_2}(b) = f_{kl_2}(c)$. Let $l_1, l_2 \leq l$; then $f_{jl}(b) = f_{l_1 l} f_{jl_1}(b) = f_{l_1 l} f_{il_1}(a) = f_{il}(a)$, and $f_{jl}(b) = f_{l_2 l} f_{jl_2}(b) = f_{l_2 l} f_{kl_2}(c) = f_{kl}(c)$, hence $f_{il}(a) = f_{kl}(c)$, so that $a \equiv c$. Put $\tilde{A} := A/\equiv$. We prove that $\tilde{A}$ can be endowed with a structure of a D-poset. Let $a, b \in A$, $a \in A_i$, $b \in A_j$ and let $\tilde{a} = a/\equiv$, $\tilde{b} = b/\equiv$ be the corresponding equivalence classes in $\tilde{A}$. We define a partial binary operation $\ominus$ on $\tilde{A}$ as follows:

$\tilde{b} \ominus \tilde{a}$ exists iff there is $k \in I$, $i, j \leq k$, and
$f_{jk}(b) \ominus f_{ik}(a)$ is defined in A_k;
then $\tilde{b} \ominus \tilde{a} = (f_{jk}(b) \ominus f_{ik}(a))^{\sim}$

To prove that the operation $\ominus$ is well defined, let a_1, b_1 be any other representants with $a_1 \in A_{i_1}$, $b_1 \in A_{j_1}$. There is $l \in I$ with $i_1, j_1, k \leq l$, and

$$f_{j_1 l}(b_1) = f_{jl}(b) = f_{kl} f_{jk}(b)$$

$$f_{i_1 l}(a_1) = f_{il}(a) = f_{kl} f_{ik}(a)$$

Now $f_{jk}(b) \ominus f_{ik}(a)$ exists in A_k implies $f_{j_1 l}(b_1) \ominus f_{i_1 l}(a_1)$ exists in A_l, and $f_{j_1 l}(b_1) \ominus f_{i_1 l}(a_1) = f_{kl}(f_{jk}(b) \ominus f_{ik}(a))$, hence $f_{j_1 l}(b_1) \ominus f_{i_1 l}(a_1) \equiv f_{jk}(b) \ominus f_{ik}(a)$.

Clearly, $\tilde{1} = \cup\{a \in A_i: \exists j \geq i, f_{ij}(a) = 1_j\}$, $\tilde{0} = \cup\{a \in A_i: \exists j \geq i, f_{ij}(a) = 0_j\}$, where 1_i and 0_i are the greatest and the smallest elements in A_i, respectively.

Define $\tilde{a} \leq \tilde{b}$ iff $\tilde{b} \ominus \tilde{a}$ exists. Let $\tilde{a} \leq \tilde{b} \leq \tilde{c}$, and $a \in A_i$, $b \in A_j$, $c \in A_k$ be any representants. There is $l \in I$ with $i, j, k \leq l$, and $f_{il}(a) \leq f_{jl}(b) \leq f_{kl}(c)$ hold in A_l. The fact that A_l is a D-poset implies that $\ominus$ is a difference operation on $\tilde{A}$. From the construction it follows that $\equiv$ restricted to A_i for each $i \in I$ is a congruence relation. For every $i \in I$, let $f_i: A_i \to \tilde{A}$ be the natural projection $f_i(a) = \tilde{a}$. If $a \in A_i$ and $i \leq j$, then for any k, $i \leq j \leq k$, $f_{ik}(a) = f_{jk} f_{ij}(a)$, so that $a \equiv f_{ij}(a)$, and hence $f_j f_{ij} = f_i$.

Let $q := ((g_i: A_i \to B, i \in I), B)$ be any system compatible with A_I. Define $g: \tilde{A} \to B$ via $g(f_i(a)) = g_i(a)$. Now $f_i(a) = f_i(b)$, $a, b \in A_i$ iff there is $j \in I, i \leq j$ with $f_{ij}(a) = f_{ij}(b)$. But then $g_i(a) = g_j f_{ij}(a) = g_j f_{ij}(b) = g_i(b)$, hence g is a well-defined morphism. Clearly, g is unique. This concludes the proof that $\tilde{A} = \lim_{\to} A_I$. ■

Corollary 2.8. If A_i, $i \in I$, in a directed system A_I are orthoalgebras (orthomodular posets) then the direct limit $\lim_{\to} A_I$ is also an orthoalgebra (orthomodular poset).

Proof. According to Navara and Pták (n.d.), a D-poset is an orthoalgebra iff

(i) $a \leq 1 \ominus a \Rightarrow a = 0$.

A D-poset is an orthomodular poset iff (i) holds along with the following:

(ii) $a \perp b, b \perp c, c \perp a$ imply $a \leq (1 \ominus c) \ominus b$.

Indeed, if A is an OMP, then $b \ominus a = b \wedge a'$ for $a \leq b$, and (ii) becomes $a \leq b' \wedge c'$. Conversely, recall that an orthoalgebra is an OMP iff $a \perp b$, $b \perp c$, $c \perp a$ imply $a \oplus b \perp c$. Now if (ii) holds, then $a = ((1 \ominus c) \ominus b) \ominus d = ((1 \ominus c) \ominus d) \ominus b$, which implies $a \oplus b = (1 \ominus c) \ominus d \leq 1 \ominus c$.

From the proof of Theorem 2.7 it can be derived that conditions (i), (ii) for the difference $\ominus$ are satisfied in $\lim_{\to} A_I$ if they are satisfied in every A_i, $i \in I$. ■

In Dvurečenskij (1994) the notions of a bimorphism and tensor product of two D-posets are introduced. These notions can be generalized to any finite number $n \in \mathbb{N}$ of D-posets in a natural way.

Definition 2.9. Let $A_1, A_2, \ldots, A_n, B$ be D-posets. A mapping $\beta: A_1 \times \cdots \times A_n \to B$ is called an *n-morphism* (or a *multimorphism* in general) iff:

(i) $a, b \in A_i$, $a \perp b$, $q_j \in A_j$, $j \neq i$, $1 \leq i, j \leq n$, imply $\beta(u_j)_{j \leq n} \perp \beta(v_j)_{j \leq n}$, where $u_j = q_j = v_j$, $j \neq i$, and $u_i = a$, $v_i = b$, and $\beta((u_j)) \oplus \beta((v_j)) = \beta((z_j))$, where $z_j = q_j$, $j \neq i$, $z_i = a_i \oplus b_i$.

(ii) $\beta(1, 1, \ldots, 1) = 1$.

Let $\beta: A_i \times A_2 \cdots \times A_n \to B$ be a multimorphism. To simplify our considerations, we introduce the following conventions.

Let $N = \{1, 2, \ldots, n\}$ be a finite sequence and $K = \{k_1, \ldots, k_m\}$, $m \leq n$, a subsequence of N. For any $a = (a_{k_1}, \ldots, a_{k_m}) \in A_{k_1} \times \cdots \times A_{k_m}$, we define the following:

(i) $a^{KN} = (a_i)_{i \leq n}$ is the element in $A_1 \times \cdots \times A_n$ such that $a_i = a_{k_j}$ for $i = k_j$ and $a_i = 1$ if $i \notin K$.

(ii) $\beta^{KN}: A_{k_1} \times \cdots \times A_{k_m} \to B$ is a mapping defined by $\beta^{KN}(a) = \beta(a^{KN})$.

Clearly, β^{KN} is a multimorphism. In particular, if $K = \{j\}$, then $\beta^{\{j\}N}: A_j \to B$ is a morphism.

Let $K := \{k_1, \ldots, k_m\}$ be a subsequence of $\{1, 2, \ldots, n\} =: N$. For any $a := (a_i)_{i \in K}$ and $b := (b_j)_{j \in N \setminus K}$ let $a^{KN,b} = (u_i)_{i \leq n}$ be such that $u_i = a_{k_j}$ for $i = k_j \in K$, $u_j = b_j$ for $j \in N \setminus K$. Clearly $a^{KN} \equiv a^{KN,1}$.

Lemma 2.10. Let $\beta: A_1 \times \cdots \times A_N \to B$ be a multimorphism. Let $N := \{1, 2, \ldots, n\}$, $K := \{k_1, \ldots, k_n\} \subseteq N$; $a = (a_i)_{i \in K}$, $b = (b_i)_{i \in K}$, $c = (c_i)_{i \in N \setminus K}$, $d = (d_i)_{i \in N \setminus K}$. If $\beta(a^{KN}) \perp \beta(b^{KN})$, then $\beta(a^{KN,c}) \perp \beta(b^{KN,d})$. In particular, if $\beta: A_1 \times A_2 \to B$ is a bimorphism, then $a \perp b (a, b \in A_1) \Rightarrow \beta(a, c) \perp \beta(b, d)$ for any $c, d \in A_2$.

Proof. We will proceed by induction of card$(N \setminus K)$.

If card$(N \setminus K) = 1$, $N \setminus K = \{j\}$, then $\beta(a^{KN}) = \beta(a^{KN,c}) \oplus \beta(a^{KN,c^\perp})$; $\beta(b^{KN}) = \beta(b^{KN,d}) \oplus \beta(b^{KN,d^\perp})$ for any $c, d \in A_j$, where $c^\perp = 1_j \ominus c$, $d^\perp = 1_j \ominus d$, and

$$\beta(a^{KN}) \oplus \beta(b^{KN}) = (\beta(a^{KN,c}) \oplus (\beta(a^{KN,c^\perp})) \oplus (\beta(b^{KN,d}) \oplus \beta(b^{KN,d^\perp}))$$

implies that $\beta(a^{KN,c}) \oplus \beta(b^{KN,d})$ exist, hence $\beta(a^{KN,c}) \perp \beta(b^{KN,d})$.

If card$(N \setminus K) = k$, we choose $i_0 \in N \setminus K$, and define $c_0 = (u_i)_{i \in N \setminus K}$, $u_{i_0} = c_{i_0}$, $u_i = 1$, $i \neq i_0$; $d_0 = (v_i)_{i \in N \setminus K}$, $v_{i_0} = d_{i_0}$, $v_i = 1$, $i \neq i_0$. By the first part of the proof, $\beta(a^{KN,c_0}) \perp \beta(b^{KN,d_0})$. Now we can replace K by $K \cup \{i_0\}$, and we obtain the desired result applying the induction hypothesis (to appropriately defined a', b', c', d' replacing a, b, c, d). ∎

Definition 2.11. Let $A_1, \ldots, A_n$ be D-posets. We say that a pair (R, τ) consisting of a difference poset R and a n-morphism $\tau: A_1 \times \cdots \times A_n \to R$ is a *tensor product* of $A_1, A_2, \ldots, A_n$ iff the following conditions are satisfied:

(i) If L is a D-poset and $\beta: A_1 \times \cdots \times A_n \to L$ is a n-morphism, there exists a morphism $\phi: R \to L$ such that $\beta = \phi \circ \tau$.
(ii) Every element of R is a finite orthogonal sum of elements of the form $\tau((a_i)_{i \leq n})$ with $a_i \in A_i$, $i \leq n$.

Clearly, if a tensor product exists, it is unique up to isomorphism.

The following statement is a generalization of Theorem 7.3 in Dvurečenskij (1994); the proof of it can be obtained by a modification of the proof of the latter theorem.

Theorem 2.12. A tensor product of D-posets $A_1, \ldots, A_n$ exists iff there is at least one difference poset L and an n-morphism $\beta: A_1 \times \cdots \times A_n \to L$.

As a corollary we obtain that a tensor product of $\mathscr{E}(H_i)$, $i = 1, 2, \ldots, n$, where $\mathscr{E}(H_i)$ is the set of all effects on the Hilbert space H_i, in the category of D-posets exists. Indeed, we take for L in Theorem 2.12 the set $\mathscr{E}(H_1 \otimes \cdots \otimes H_n)$ of all effects on the Hilbert space tensor product $H_1 \otimes \cdots \otimes H_n$.

We note that if in a class of D-posets a tensor product of any two elements exists, then a tensor product of n elements exists, too. This can be proved by induction according to the following considerations.

Let $A_1, \ldots, A_n$ be D-posets from the considered class. Let (B, β) be the tensor product of $A_1, \ldots, A_{n-1}$, and let (C, γ) be the tensor product of B, A_n. Define $\tilde{\gamma}: A_1 \times \cdots \times A_{n-1} \times A_n \to C$ by $\tilde{\gamma}(a_1, \ldots, a_{n-1}, a_n) = \gamma((\beta(a_1, \ldots, a_{n-1}), a_n)$. Clearly, $\tilde{\gamma}$ is an n-morphism. We claim that $(C, \tilde{\gamma})$ is a tensor product of $A_1, \ldots, A_{n-1}, A_n$. It is easily seen that every element in C is a finite orthogonal sum of elements of the form $\gamma(\beta(a_1, \ldots, a_{n-1}), a_n) = \tilde{\gamma}(a_1, \ldots, a_{n-1}, a_n)$. Let $\kappa: A_1 \times \cdots \times A_{n-1} \times A_n \to D$ be an n-morphism into a D-poset D in the same class. Put $N := \{1, \ldots, n\}$, $K := \{1, \ldots, n-1\}$. Then $\kappa^{KN}: A_1 \times \cdots \times A_{n-1} \to D$ is an $(n-1)$-morphism. Therefore, there exists a morphism $\phi: B \to D$ with $\phi \circ \beta = \kappa^{KN}$. Now define a mapping $\tilde{\kappa}$ by $\tilde{\kappa}(\beta(a_1, \ldots, a_{n-1}), a_n) = \kappa(a_1, \ldots, a_{n-1}, a_n)$. If $\beta(a_1, \ldots, a_{n-1}) \perp \beta(b_1, \ldots, b_{n-1})$, then $\phi \circ \beta(a_1, \ldots, a_{n-1}) \perp \phi \circ \beta(b_1, \ldots, b_{n-1})$; hence

$$\kappa^{KN}(a_1, \ldots, a_{n-1}) \perp \kappa^{KN}(b_1, \ldots, b_{n-1})$$

and Lemma 2.10 implies that

$$\begin{aligned} \kappa(a_1, \ldots, a_{n-1}, a_n) &= \tilde{\kappa}(\beta(a_1, \ldots, a_{n-1}), a_n) \\ \perp \kappa(b_1, \ldots, b_{n-1}, b_n) &= \tilde{\kappa}(\beta(b_1, \ldots, b_{n-1}), b_n) \end{aligned}$$

Hence $\tilde{\kappa}$ can be extended to a bimorphism from $B \times A_n$ to D. Then there is a morphism $\psi: C \to D$ such that $\psi \circ \gamma = \tilde{\kappa}$, and

$$\begin{aligned} \psi \circ \tilde{\gamma}(a_1, \ldots, a_{n-1}, a_n) &= \psi \circ \gamma(\beta(a_1, \ldots, a_{n-1}), a_n) \\ &= \tilde{\kappa}(\beta(a_1, \ldots, a_{n-1}), a_n) = \kappa(a_1, \ldots, a_{n-1}, a_n) \end{aligned}$$

i.e., $\psi \circ \tilde{\gamma} = \kappa$. This proves the universal property of $(C, \tilde{\gamma})$.

In what follows, we will often denote a tensor product $(R \otimes)$ of $A_1, \ldots, A_n$ by $R = A_1 \otimes \cdots \otimes A_n$, and write $a_1 \otimes \cdots \otimes a_n$ instead of $\otimes(a_1, \ldots, a_n)$.

Theorem 2.13. Let $A_1, \ldots, A_n$ be D-posets and let the tensor product $A_1 \otimes \cdots \otimes A_n$ exist. Then:

(i) For any $K = \{k_1, \ldots, k_m\} \subseteq \{1, 2, \ldots, n\} = N$, a tensor product $A_{i_1} \tilde{\otimes}_m \cdots \tilde{\otimes}_m A_{k_m}$ of $A_{k_1}, \ldots, A_{k_m}$ exists.

(ii) For any $1 \leq m \leq n$, $A_1 \otimes \cdots \otimes A_n = (A_1 \otimes_m \cdots \otimes_m A_m) \otimes_2 (A_{m+1} \otimes_{n-m} \cdots \otimes_{n-m} A_n)$, where $\otimes_m$ denotes the m-morphism in the tensor product of $A_1, \ldots, A_m$, and $\otimes_{n-m}$ denotes the $(n - m)$-morphism in the tensor product of $A_{m+1}, \ldots, A_n$.

Proof. Item (i) is a direct consequence of Theorem 2.12 and the fact that $\otimes^{KN}$: $A_{k_1} \times \cdots \times A_{k_m} \to A_1 \otimes \cdots \otimes A_n$, $\otimes^{KN}(a_{k_1}, \ldots, a_{k_m}) = \otimes_m((a_{k_i}, \ldots, a_{k_m}{}^{KN})$ is an m-morphism.

(ii) It can be proved by induction using similar methods as in the remarks preceding Theorem 2.13. ∎

Proposition 2.14. Let T be any set, and let $(\mathcal{T}, \subseteq)$ be a directed poset of finite subsequences of elements of T directed by inclusion. For every $t \in T$, let A_t be a D-poset. Assume that for every $F \in \mathcal{T}$, $F = \{t_1, \ldots, t_n\}$, $(A_F, \otimes_F)$ is a tensor product of $A_{t_1}, \ldots, A_{t_n}\}$. Then for every F, $G \in \mathcal{T}$, $F \subset G$ there is a morphism f_{FG}: $A_F \to A_G$ such that (A_F, f_{FG}) is a directed system.

Proof. Let F, $G \in \mathcal{T}$, $F \subseteq G$,

$$G = \{a_{t_1}, \ldots, a_{t_n}\}, \qquad F = \{a_{t_{k_1}}, \ldots, a_{t_{k_m}}\}, \qquad m \le n$$

Let $(A_F, \otimes_F)$, $(A_G, \otimes_G)$ be the corresponding tensor products. Then

$$\otimes_G^{FG}: A_{t_{k_1}} \times \cdots \times A_{t_{k_m}} \to A_G$$

is a multimorphism; therefore there is a (unique) morphism f_{FG}: $A_F \to A_G$ such that $f_{FG} \circ \otimes_F = \otimes_G^{FG}$. Clearly $f_{FF} = id_F$ (where id_F denotes identity on A_F).

Now let $F, G, H \in \mathcal{T}$, $F \subseteq G \subseteq H$, $F = \{A_{t_1}, \ldots, A_{t_k}\}$. Then we obtain, for any $a \in A_{t_1} \times \cdots \times A_{t_k}$, $f_{GH} \circ f_{FG} \circ \otimes_F(a) = f_{GH}(f_{FG} \circ \otimes_F(a)) = f_{GH}(\otimes_G^{FG}(a)) = f_{GH}(\otimes_G(a^{FG})) = \otimes_H((a^{FG})^{GH}) = \otimes_H(a^{FH}) = \otimes_H^{FH}(a)$, hence $f_{GH} \circ f_{FG} = f_{FH}$. ∎

As a consequence, a direct limit of $(A_F, f_{FG}, F, G \in \mathcal{T}, F \le G)$ exists.

An example of tensor products in a special category of D-posets can be obtained as follows (Dvurečenskij and Pulmannová, 1994-*a*,*b*).

Let $I = [0, 1]$ be endowed with the natural ordering and the difference $b \ominus a = b - a$, $a, b \in I$. Then I is a D-poset. Any homomorphism m from a D-poset A to I is called a *state* on A. A set $\mathcal{P}$ of states on A is:

(i) *separating* if $m(a) = m(b)$ for any $m \in \mathcal{P}$ implies $a = b$.

(ii) *full* iff $\mathcal{P}$ is separating and if $\Sigma_{i=1}^n m(a_i) = 1$ for any $m \in \mathcal{P}$, then $\oplus_{i=1}^n a_i$ exists in A and $\oplus_{i=1}^n a_i = 1$.

(iii) *order determining* iff the condition $m(a) \le m(b)$ for all $m \in \mathcal{P}$ implies $a \le b$.

If $\mathcal{P}$ is order determining, then $\mathcal{P}$ is full (Dvurečenskij and Pulmannová, 1994-*b*).

Let $(A_i)_{i \le n}$ be D-posets. We say that a couple $\mathcal{K} = (\mathcal{L}, \mathcal{B})$, where $\mathcal{L}$ is a nonvoid family of D-posets and $\mathcal{B}$ is a nonvoid family of n-morphisms on

$A_1 \times \cdots \times A_n$ such that (i) for any $\beta \in \mathcal{B}$ there exists a D-poset $L \in \mathcal{L}$ such that $\beta: A_1 \times \cdots \times A_n \to L$, and (ii) for any $L \in \mathcal{L}$ there is an n-morphism $\beta \in \mathcal{B}$ with $\beta: A_1 \times \cdots \times A_n \to L$, is said to be a *consistent class* for $(A_i)_{i\leq n}$. In what follows, $\Pi_{i\leq n} A_i := A_1 \times \cdots \times A_n$.

Definition 2.15. Let $\mathcal{K} = (\mathcal{L}, \mathcal{B})$ be a consistent class for the D-posets A_i, $i \leq n$. We say that a pair (R, τ) consisting of a difference poset R and an n-morphism $\tau: \Pi_{i\leq n} A_i \to R$ is a *tensor product of* A_i, $i \leq n$, *in the class* $\mathcal{K} = (\mathcal{L}, \mathcal{B})$ iff the following conditions are satisfied:

(i) $R \in \mathcal{L}$, $\tau \in \mathcal{B}$.
(ii) If L is a D-poset in $\mathcal{L}$, and β is a bimorphism in $\mathcal{B}$, $\beta: \Pi_{i\leq n} A_i \to L$, there exists a morphism $\phi: R \to L$ such that $\beta = \phi \circ \tau$.
(iii) Every element of R is a finite $\oplus$-orthogonal sum of elements of the form $\tau((a_i)_{i\leq n})$ with $a_i \in A_i$, $i \leq n$.

Similarly as above, if a tensor product (R, τ) of A_i, $i \leq n$, exists in the class $\mathcal{K}$, it is unique up to an isomorphism. It is clear that if $\mathcal{L}$ consists of all D-posets L for which there exists a bimorphism $\beta: \Pi_{i\leq n} A_i \to L$ and $\mathcal{B}$ is the family of all those bimorphisms, then tensor products from Definitions 2.11 and 2.15 coincide.

Suppose that $\mathcal{P}_i$ are nonempty families of states on difference posets A_i, $i \leq n$. We set $\mathcal{P} := \Pi_{i\leq n} \mathcal{P}_i$ and, if $\lambda = (\lambda_i)_{i\leq n} \in \mathcal{P}$ and $(a_i)_{i\leq n} \in \Pi_{i\leq n} A_i$, then $\lambda((a_i)_{i\leq n}) := \lambda_1(a_1) \cdots \lambda_n(a_n)$.

Theorem 2.16. Let $\mathcal{P}_i$, $i \leq n$, be nonvoid systems of states on the D-posets P_i, $i \leq n$, respectively, $\mathcal{P} = \Pi_{i\leq n} \mathcal{P}_i$. Let $\mathcal{L}_{\mathcal{P}}$ be the set of all D-posets L such that there is an n-morphism $\kappa: P_1 \times \cdots \times P_n \to L$, and the set $\mathcal{P}_\kappa := \{\mu_1 \otimes \cdots \otimes \mu_n: \mu_i \in \mathcal{P}_i, i \leq n\}$ is a full system of states on L, where $\mu_1 \otimes \cdots \otimes \mu_n(\kappa((a_i)_{i\leq n}) := \mu_1(a_1) \cdots \mu_n(a_n)$, $a_i \in P_i$, and let $\mathcal{B}_{\mathcal{P}}$ be the set of all these n-morphisms κ's. Then $\mathcal{K} = (\mathcal{L}_{\mathcal{P}}, \mathcal{B}_{\mathcal{P}})$ is a consistent class for P_i, $1 \leq i \leq n$, and there exists a tensor product of P_i, $1 \leq i \leq n$.

Proof. Let X be the subset of $\Pi_{i\leq n} A_i$ consisting of all n-tuples $(a_i)_{i\leq n}$ with $a_i \neq 0$, $\forall i$. If $M = ((a_i^1)_{i\leq n}, \ldots, (a_i^k)_{i\leq n})$ is a finite sequence of elements from X and $\lambda \in \mathcal{P}$, we put

$$\lambda(M) = \sum_{j=1}^{k} \lambda((a_i^j)_{i\leq n})$$

with the understanding that if $M = \emptyset$, then $\lambda(M) = 0$.

Define now the set $\mathcal{F}$ of all finite sequences $T = ((a_i^j)_{i\leq n})_{j\leq k}$ such that $\lambda(T) = 1$ for any $\lambda \in \mathcal{P}$. Since $\lambda((1, \ldots, 1)) = 1$, $\mathcal{F}$ is nonvoid. It is easy to see that for any $(a_i)_{i\leq n} \in X$ there is a finite sequence from $\mathcal{F}$ containing $(a_i)_{i\leq n}$.

Denote by $\mathscr{E}(\mathscr{F})$ the set of all finite sequences $((a_i^j)_{i\le n})_{j\in J}$ such that $J \subseteq I$ and $((a_i)_{i\le n})_{j\in I} \in \mathscr{F}$. We put $((a_i^j)_{i\le n})_{j\in\emptyset} = 0$.

For $A, B \in \mathscr{E}(\mathscr{F})$ we define $A \approx B$ iff $\lambda(A) = \lambda(B)$ for any $\lambda \in \mathscr{P}$. Then $\approx$ is an equivalence on $\mathscr{E}(\mathscr{F})$, and let $\pi(a) := \{B \in \mathscr{E}(\mathscr{F}): B \approx A\}$. Let $\Pi(X) := \{\pi(A): A \in \mathscr{E}(\mathscr{F})\}$. We organize $\Pi(X)$ into a poset by defining a partial order $\le$ on $\Pi(X)$ as follows: $\pi(A) \le \pi(B)$, where $A = ((a_i^j)_{i\le n})_{j\le k}$, $B = ((b_i^j)_{i\le n})_{j\le m}$, iff there is $C = ((c_i^{j'})i \le n)_{j'\le s} \in \mathscr{E}(\mathscr{F})$ such that $M := ((a_i^j)_{i\le n})_{j\le k} \cup ((c_i^{j'})_{i\le n})_{j'\le s} \in \mathscr{E}(\mathscr{F})$ and $\lambda(M) = \lambda(B)$ for any $\lambda \in \mathscr{P}$. Then $\pi(\emptyset)$ and $\pi(T)$, where $T \in \mathscr{F}$ is arbitrary, are the smallest and greatest elements in $\Pi(X)$.

The difference operation $\ominus$ on $\Pi(X)$ is defined whenever $\pi(A) \le \pi(B)$, and $\pi(B) \ominus \pi(A) = \pi(C)$, where A, B, C satisfy the above conditions for the partial ordering $\le$. Then $\ominus$ is defined correctly, and $\Pi(X)$ is a difference poset.

Define a mapping κ_0: $\prod_{i\le n} A_i \to \Pi(X)$ via

$$\kappa_0((a_i)_{i\le n}) = \begin{cases} \pi((a_i)_{i\le n}), & (a_i)_{i\le n} \in X \\ 0, & (a_i)_{i\le n} \notin X \end{cases}$$

Then κ_0 is, evidently, an n-morphism. From the construction of $\Pi(X)$ we see that any $u \in \Pi(X)$ is of the form $u = \bigoplus_{j\le k} \kappa_0((a_i^j)_{i\le n})$, and the mapping $(\otimes_{\kappa_0})_{i\le n}\, \mu_i$, $\mu_i \in \mathscr{P}_i$, $i \le n$, on $\Pi(X)$ defined by

$$(\otimes_{\kappa_0})_{i\le n}\, \mu_i(\kappa_0((a_i)_{i\le n}) = \mu_1(a_1) \cdots \mu_n(a_n), \qquad a_i \in A_i, \quad i \le n$$

is a state on $\Pi(X)$. In addition, $\mathscr{P}_{\kappa_0}$ is a full system of states on $\Pi(X)$. Therefore, $\mathscr{L}_{\mathscr{P}} \ne \emptyset$ and let $\mathscr{B}_{\mathscr{P}}$ be the set of all n-morphisms κ such that κ maps $\Pi_{i\le n} A_i$ into some $L \in \mathscr{L}_{\mathscr{P}}$ and $\mathscr{P}_\kappa$ be a full system of states on L. Then $\mathscr{K}_{\mathscr{P}} = (\mathscr{L}_{\mathscr{P}}, \mathscr{B}_{\mathscr{P}})$ is a consistent class for A_i, $i \le n$.

We claim that $(\Pi(X), \kappa_0)$ is a tensor product of A_i, $i \le n$, in the class $\mathscr{K}_{\mathscr{P}}$. Choose $L \in \mathscr{L}_{\mathscr{P}}$ and an n-morphism κ: $\Pi_{i\le n} A_i \to L$. Since $\mathscr{P}_\kappa$ is full for L, it follows that if $\kappa_0((a_i)_{i\le n}) = \kappa_0((a_i')_{i\le n})$, then $\kappa((a_i)_{i\le n}) = \kappa((a_i')_{i\le n})$, and we can define a mapping ϕ such that $\phi(\kappa_0((a_i)_{i\le n}) = \kappa((a_i)_{i\le n})$, $a_i \in A_i$, $i \le n$. We claim that we can extend ϕ to the whole $\Pi(X)$ via $\phi(u) = \bigoplus_{j\le k} \kappa((a_i^j)_{i\le n})$, whenever $u = \bigoplus_{j\le k} \kappa_0((a_i^j)_{i\le n})$, to a well-defined multimorphism. Indeed, let $u = \bigoplus_{j\le k} \kappa_0((a_i^j)_{i\le n}) = \bigoplus_{l\le m} \kappa_0((b_i^l)_{i\le n}) = v$. Then $u^\perp$ has the form $u^\perp = \bigoplus_{q\le s} \kappa_0((c_i^q)_{i\le n})$ and for all $\mu_i \in \mathscr{P}_i$, $i \le n$, we have

$$\begin{aligned} 1 &= (\otimes_{\kappa_0})_{i\le n}\, \mu_i(u \oplus u^\perp) \\ &= (\otimes_{\kappa_0})_{i\le n}\, \mu_i(u) + (\otimes_{\kappa_0})_{i\le n}\, \mu_i(u^\perp) \\ &= (\otimes_{\kappa_0})_{i\le n}\, \mu_i(v) + (\otimes_{\kappa_0})_{i\le n}\, \mu_i(u^\perp) \end{aligned}$$

$$= \sum_{j\le k} \prod_{i\le n} \mu_i(a_i^j) + \sum_{q\le s} \prod_{i\le n} \mu_i(c_i^q)$$

$$= \sum_{l\le m} \prod_{i\le n} \mu_i(b_i^l) + \sum_{q\le s} \prod_{i\le n} \mu_i(c_i^q)$$

$$= \sum_{j\le k} (\otimes_\kappa)_{i\le n}\, \mu_i(\kappa((a_i^j)_{i\le n}) + \sum_{q\le s} (\otimes_\kappa)_{i\le n}\, \mu_i(\kappa(c_i^q)_{i\le n})$$

$$= \sum_{l\le m} (\otimes_\kappa)_{i\le n}\, \mu_i(\kappa((b_i^l)_{i\le n}) + \sum_{q\le s} (\otimes_\kappa)_{i\le n}\, \mu_i(\kappa(c_i^q)_{i\le n})$$

Hence

$$\bigoplus_{k\le m} \kappa((a_i^j)_{i\le n}) \oplus \bigoplus_{q\le s} \kappa((c_i^q)_{i\le n}) = 1 = \bigoplus_{l\le m} \kappa((b_i^l)_{i\le n}) \oplus \bigoplus_{q\le s} \kappa((c_i^q)_{i\le n})$$

so that

$$\bigoplus_{j\le k} \kappa((a_i^j)_{i\le n})) = \bigoplus_{l\le m} \kappa((b_i^l)_{i\le n})$$

It is easy to check that ϕ is the morphism in question, which proves the assertion of the theorem. ■

The tensor product $((\otimes_{\mathcal{P}})_{i\le n} A_i, \otimes_{\mathcal{P}}) := (\Pi(X), \kappa_0)$ of the D-posets A_i, $i \le n$, in the class $\mathcal{K}_{\mathcal{P}}$ is said to be a *state tensor product* of A_i, $i \le n$, *with respect to the state system* $\mathcal{P} = \Pi_i\, \mathcal{P}_i$. It is easy to check that an analog of Theorem 2.13 holds for state tensor products. Also, if $\mathcal{P}_i$ is order determining for A_i for each $i \le n$, then $\otimes_{\mathcal{P}}$ is a closed multimorphism.

Let $\mathcal{E}(H_i)$, $i \le n$, be sets of all effects on the Hilbert spaces H_i, $i \le n$, respectively. Then they have order-determining sets of states. As a direct generalization of Dvurečenskij and Pulmannová (1994-*b*) we obtain that the state tensor product consists of all elements of the form $\Sigma_k\, E_1^k \otimes \cdots \otimes E_n^k$, where $E_i \in \mathcal{E}(H_i)$, $i \le n$, $\otimes$ is the usual tensor product of operators, for which there are $F_1^j \otimes \cdots \otimes F_n^j$ such that $\Sigma_k\, E_1^k \otimes \cdots \otimes E_n^k \oplus \Sigma_j\, F_1^j \otimes \cdots \otimes F_n^j = I_1 \otimes \cdots \otimes I_n$ (all summations are over finite index sets).

Let A_t, $t \in T$, be a system of D-posets such that for each $t \in T$, there is an order-determining set $\mathcal{P}_t$ of states. Then a directed system constructed from state tensor products of these D-posets over finite subsets of T has the property that every morphism f_{EF}, $E \subseteq F$, is closed. This can be easily derived from the construction of the state tensor products.

3. GENERALIZED AXIOMS FOR A SPACE OF HISTORIES

Gell-Mann and Hartle axioms postulated a new approach to quantum theory in which the notion of "history" is ascribed a fundamental role, i.e., a history may be an irreducible entity in its own right that is not necessarily to be constructed as a time-ordered string of single-time propositions. According to Isham (n.d.), these axioms and definitions are assentially as follows:

1. The fundamental ingredients in the theory are a space of *histories* and a space of *decoherence functionals*, which are complex-valued functions of pairs of histories.

2. The set of histories possesses a partial order $\leq$. If $\alpha \leq \beta$, then β is said to be *coarser* than α, or a *coarse-graining* of α; dually, α is *finer* than β, or a *fine-graining* of β. Heuristically this means that α possesses a more precise specification than β.

3. There is a notion of two histories α, β to be *disjoint*, $\alpha \perp \beta$. Heuristically, if $\alpha \perp \beta$, then if either α or β is "realized," the other is "excluded."

4. There is a *unit* history 1 (heuristically, the history that is always realized) and a *null* history 0 (heuristically, the history that is never realized). For all histories we have $0 \leq \alpha \leq 1$.

5. Two histories, α, β that are disjoint can be combined to form a new history $\alpha \vee \beta$ (heuristically, the history "α or β").

6. A set of histories $\alpha^1, \alpha^2, \ldots, \alpha^N$ is said to be *exclusive* if $\alpha^i \perp \alpha^j$ for all $i, j = 1, 2, \ldots, N$. The set is *exhaustive* (or *complete*) if it is exclusive and if $\alpha^1 \vee \cdots \vee \alpha^N = 1$.

7. Any decoherence functional d satisfies the following conditions:

(a) $d(0, \alpha) = 0$ for all α.
(b) Hermicity: $d(\alpha, \beta) = \bar{d}(\beta, \alpha)$ for all α, β.
(c) Positivity: $d(\alpha, \alpha) \geq 0$ for all α.
(d) Additivity: if $\alpha \perp \beta$, then, for all γ, $d(\alpha \vee \beta) = d(\alpha, \gamma) + d(\beta, \gamma)$.
(e) Normalization: If $\alpha^1, \alpha^2, \ldots, \alpha^N$ and $\beta^1, \beta^2, \ldots, \beta^M$ are two complete sets of histories then

$$\sum_{i=1}^{N} \sum_{j=1}^{M} d(\alpha^i, \beta^j) = 1$$

It is important to note that this axiomatic scheme is given a physical interpretation only in relation to *consistent* sets of histories. A complete set C of histories is said to be (strongly) consistent with respect to a particular decoherence functional d if $d(\alpha, \beta) = 0$ for all $\alpha, \beta \in C$ such that $\alpha \neq \beta$. Under these circumstances, $d(\alpha, \alpha)$ is given the physical interpretation as the *probability* that the history α will be "realized." The Gell-Mann and Hartle axioms then guarantee that the usual Kolmogoroff probability sum rules will be satisfied.

Let us briefly summarize how "histories" are understood in the conventional interpretation of an open Hamiltonian quantum system that is subject to measurement by an external (classical) observer.

Let $U(t_1, t_0)$ denote the unitary time-evolution operator from time t_0 to t_1, i.e., $U(t_1, t_0) = \exp\}-i(t_1 - t_0)H/\hbar\}$. Then, in the Schrödinger picture, the density operator state $\rho(t_0)$ at time t_0 evolves in time $t_1 - t_0$ to $\rho(t_1)$, where

$$\rho(t_1) = U(t_1, t_0)\rho(t_0)U(t_1, t_0)^\dagger = U(t_1, t_0)\rho(t_0)U(t_1, t_0)^{-1} \tag{3.1}$$

Suppose that a measurement is made at time t_1 of a property represented by a projection operator P. Then the probability that the property will be found is

$$\begin{aligned}\mathrm{Prob}(P = 1; \rho(t_1)) &= \mathrm{tr}(P\rho(t_1)) \\ &= \mathrm{tr}(PU(t_1, t_0)\rho(t_0)U(t_1, t_0)^\dagger) \\ &= \mathrm{tr}(P(t_1)\rho(t_0))\end{aligned} \tag{3.2}$$

where

$$P(t_1) := U(t_1, t_0)^\dagger P(t_0)U(t_1, t_0) \tag{3.3}$$

is, in the Heisenberg picture, an operator defined with respect to the time t_1. If the result of this measurement is kept, then according to the von Neumann–Lüders "reduction" postulate, the appropriate density matrix to use for any future calculation is

$$\rho_{\mathrm{red}}(t_1) := \frac{P(t_1)\rho(t_0)P(t_1)}{\mathrm{tr}(P(t_1)\rho(t_0))} \tag{3.4}$$

Now suppose that a measurement is performed of a second observable Q at time $t_2 > t_1$. Then, according to the above, the *conditional probability* of getting $Q = 1$ at time t_2 given that $P = 1$ was found at time t_1 [and that the original state was $\rho(t_0)$] is

$$\begin{aligned}\mathrm{Prob}(Q = 1 | P = 1 \text{ at } t_1; \rho(t_0)) &= \mathrm{tr}(Q(t_2)\rho_{\mathrm{red}}(t_1)) \\ &= \frac{\mathrm{tr}(Q(t_2)P(t_1)\rho(t_0)P(t_1))}{\mathrm{tr}(P(t_1)\rho(t_0))}\end{aligned} \tag{3.5}$$

The probability of getting $P = 1$ at t_1 and $Q = 1$ at t_2 is this conditional probability multiplied by $\mathrm{Prob}(P = 1; \rho(t_1))$, i.e.,

$$\mathrm{Prob}(P = 1 \text{ at } t_1 \text{ and } Q = 1 \text{ at } t_2; \rho(t_0)) = \mathrm{tr}(Q(t_2)P(t_1)\rho(t_0)P(t_1)) \tag{3.6}$$

Generalizing to a sequence of measurements of propositions $\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}$ at times $t_1, t_2, \ldots, t_n$, the joint probability of finding all the associated properties is

$$\text{Prob}(\alpha_{t_1} = 1 \text{ at } t_1 \text{ and } \alpha_{t_2} = 1 \text{ at } t_2 \text{ and} \ldots \alpha_{t_n} = 1 \text{ at } t_n; \rho(t_0))$$
$$= \text{tr}(\alpha_{t_n}(t_n) \cdots \alpha_{t_1}(t_1)\rho(t_0)\alpha_{t_1}(t_1) \cdots \alpha_{t_n}(t_n)) \tag{3.7}$$

where we used the relation $P^2 = P$ for a projection operator.

The main assumption of the consistent histories interpretation of quantum theory is that, under appropriate conditions, the probability assignment (3.7) is still meaningful for a closed system, with no external observer or associated measurement-induced state-vector reductions. The satisfaction or otherwise of these conditions is determined by the behavior of the decoherence functional $d_\rho(\alpha, \beta)$, which, for the pair of sequences of projection operators $\alpha := (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ and $\beta := (\beta_{t_1}, \beta_{t_2}, \ldots, \beta_{t_n})$, is defined as

$$d_\rho(\alpha, \beta) := \text{tr}(C_\alpha \rho(t_0) C_\beta^\dagger) \tag{3.8}$$

where

$$C_\alpha := \alpha_{t_n}(t_n) \cdots \alpha_{t_2}(t_2)\alpha_{t_1}(t_1)$$
$$= U(t_0, t_n)\alpha_{t_n}U(t_n, t_{n-1}) \cdots U(t_3, t_2)\alpha_{t_2}U(t_2, t_1)\alpha_{t_1}U(t_1, t_0) \tag{3.9}$$

We note that the definition (3.8) satisfies the conditions 7(a)–(e) of a decoherence functional.

Isham (n.d.) suggested to find, in the quantum logic approach, candidates for the "history analogs" of the standard Hamiltonian theory. To this aim, a logic L of single-time propositions was considered. A *history filter* was defined to be any finite sequence $(\alpha_{t_1}, \ldots, \alpha_{t_n})$ of single-time propositions $\alpha_{t_i} \in L$ which is time-ordered in the sense that $t_1 < t_2 < \cdots < t_n$. Thus, in the special case when L is identified with the lattice $L(H)$ of projection operators on a Hilbert space H, a history filter concides with the notion of a "homogeneous history" in the Gell-Mann and Hartle approach. Also, it is a time-labeled version of what Mittelstaedt and Stachow (1983; see also Mittelstaedt, 1977, 1983) call a sequential conjunction, i.e., it corresponds to the proposition "α_{t_1} is true at time t_1 and then α_{t_2} is true at time t_2 and then ... and then α_{t_n} is true at time t_n." The phrase "history filter" is intended to capture the idea that each single-time proposition α_{t_i} in the collection $(\alpha_{t_1}, \ldots, \alpha_{t_n})$ serves to "filter out" the properties of the system that are realized in the history of the universe.

It is important to be able to manipulate history filters at different sets of time points. To this end, it is useful to think of a history filter as something that is defined at every time point but which is "active" only at a finite subset of points. This can be realized mathematically (Isham, n.d.) by functions for the space points T (the real line $\mathbb{R}$) to the logic L with the property that each map is equal to the unit single-time proposition for all but a finite set of t

values. Also, we will consider all history filters containing the null single-time proposition at a time t as equivalent to a null history filter, which has a null single-time proposition at all points $t \in T$ and is appended to the history filter space. It is clear that the temporal properties of a history filter are encoded in the finite set of time points at which it is active, i.e., the points $t \in T$ such that $\alpha_t \neq 1$. Let the set of all history filters be denoted by $\mathcal{U}(L)$. The set of $t \in T$ for which $\alpha_t \neq 1$ is called the *temporal support*, or just *support* of $\alpha \in \mathcal{U}(L)$, and is denoted by $\sigma(\alpha)$. The null history filter has, by definition, a null support $\Diamond$.

The set of all possible temporal supports will be denoted by $\mathcal{S}$; in our case it is just the set of all finite subsequences $\{t_1, \ldots, t_n\}$, $t_1 < t_2 < \cdots < t_n$ of $T = \mathbb{R}$.

The space $\mathcal{S}$ of supports can be equipped with the structure of a partial semigroup by saying that the support $s_2 := (t'_1, t'_2, \ldots, t'_m)$ *follows* the support $s_1 := (t_1, t_2, \ldots, t_n)$ if $t_n < t'_1$ and then defining the composition as

$$s_1 \circ s_2 = (t_1, \ldots, t_n, t'_1, \ldots, t'_m)$$

The set $\mathcal{U}(L)$ can be endowed with the structure of partial semigroup as well if we define $\circ$ on $\mathcal{U}(L)$ by the following rules: for $\alpha, \beta \in \mathcal{U}(L)$, $\alpha = (\alpha_{t_1}, \ldots, \alpha_{t_n})$, $\beta = (\beta'_{t_1}, \ldots, \beta'_{t_m})$, $\alpha \circ \beta$ is defined iff $\sigma(\alpha) < \sigma(\beta)$ and $\alpha \circ \beta = (\alpha_{t_1}, \ldots, \alpha_{t_n}, \beta'_{t_1}, \ldots, \beta'_{t_m})$. Clearly, $\sigma(\alpha \circ \beta) = \sigma(\alpha) \circ \sigma(\beta)$.

As a matter of convention, we define the null support to follow and precede every element $s \in \mathcal{S}$, so that $\alpha \circ 0$, $0 \circ \alpha$, $\alpha \circ 1$, $1 \circ \alpha$ are defined for all $\alpha \in \mathcal{U}(L)$ with values $\alpha \circ 0 = 0 \circ \alpha = 0$, $\alpha \circ 1 = 1 \circ \alpha = \alpha$, respectively. Thus the unit history 1 serves as a unit element for the semigroup structure and the null history 0 is an absorbing element.

In standard quantum theory, a natural "and" operation on a pair of history filters $\alpha := (\alpha_{t_1}, \ldots, \alpha_{t_n})$ and $\beta := (\beta_{t_1}, \ldots, \beta_{t_n})$ can be defined by

$$\alpha \wedge \beta := ((\alpha \wedge \beta)_{t_1}, \ldots, (\alpha \wedge \beta)_{t_n})$$

where $(\alpha \wedge \beta)_t = \alpha_t \wedge \beta_t$ is the "and" operation on the lattice $L = L(H)$.

In case that the quantum logic L is not a lattice, the operation "and" can be only partially defined, i.e., $\alpha \wedge \beta$ is defined iff $\alpha_{t_i} \wedge \beta_{t_i}$ is defined for every $i = 1, 2, \ldots, n$, in which case

$$\alpha \wedge \beta := ((\alpha \wedge \beta)_{t_1}, \ldots, (\alpha \wedge \beta)_{t_n})$$

For example, if β follows α, then $\alpha \circ \beta = \alpha \wedge \beta$.

Summarizing, we obtain the following modified axioms of Isham:

H1. The Space of History Filters. The fundamental ingredient in a theory of histories is a space $\mathcal{U}$ of *history filters* or possible universes. This space has the following structure.

1. $\mathcal{U}$ is a partially ordered set with a *unit* history filter 1 and a *null* history filter 0 such that $0 \leq \alpha \leq 1$ for all $\alpha \in \mathcal{U}$.
2. $\mathcal{U}$ has a partial meet operation $\wedge$ so that $1 \wedge \alpha = \alpha$ for all $\alpha \in \mathcal{U}$ and $0 \wedge \alpha = 0$ for all $\alpha \in \mathcal{U}$.
3. $\mathcal{U}$ is a partial semigroup with composition law denoted $\circ$. If α, $\beta \in \mathcal{U}$ can be combined to give $\alpha \circ \beta \in \mathcal{U}$, we say that β *follows* α, α *procedes* β, and write $\alpha \lhd \beta$. If $\alpha \circ \beta$ is defined, then $\alpha \circ \beta = \alpha \wedge \beta$.
4. The null and unit histories can always be combined with any history filter α to give

$$\alpha \circ 1 = 1 \circ \alpha = \alpha, \qquad \alpha \circ 0 = 0 \circ \alpha = 0$$

H2. The Space of Temporal Supports. Any quasitemporal properties are encoded in a partial semigroup $\mathcal{S}$ of supports with unit $\diamondsuit$. The support space has the following properties:

1. There is a semigroup homomorphism $\sigma: \mathcal{U} \to \mathcal{S}$ that assigns a support to each history filter. The support of 0 and 1 is defined to be $\diamondsuit \in \mathcal{S}$.
2. A history filter α is *nuclear* if it has no nontrivial decomposition of the form $\alpha = \beta \circ \gamma$, $\beta, \gamma \in \mathcal{U}$; a temporal support is *nuclear* if it has no nontrivial decomposition in the form $s = s_1 \circ s_2$, s_1, $s_2 \in \mathcal{S}$. Nuclear supports are the analogs of points of time; nuclear filters are the analogs of single-time propositions. A decomposition of $\alpha \in \mathcal{U}$ as $\alpha = \alpha^1 \circ \alpha^2 \circ \cdots \circ \alpha^N$ is *irreducible* if the history filters α^i, $i = 1, 2, \ldots, N$, are nuclear.

H3. The Space of History Events. The set of all histories $\mathcal{H}$ is a D-poset generated by the set of all homogeneous histories. That is, $\mathcal{H}$ is a partially ordered set with 0 and 1 as the least and greatest elements, respectively, with a partially defined binary operation $\ominus$ such that $\beta \ominus \alpha$ is defined iff $\alpha \leq \beta$ and (i) and (ii) of Remark 2.2 are satisfied. We say that histories α and β are *disjoint* (or *orthogonal*) if there is a history γ such that $\alpha = \gamma \ominus \beta$. We then write $\alpha \oplus \beta = \gamma$, and $\alpha \oplus \beta$ means, intuitively, "α or β."

We note that $\alpha \oplus \beta$ is always a coarse graining of α and β, but it need not be the supremum of α and β. Similarly, $\beta \ominus \alpha$ need not coincide with the infimum of β and $\alpha^{\perp}$. The equality $\beta = \alpha \oplus (\beta \ominus \alpha)$ holds whenever $\alpha \leq \beta$ is an analog of the orthomodular law.

Let us assume that history filters corresponding to nuclear temporal supports are elements of a D-poset L such that tensor products of any finite number of copies of L exist. The nuclear temporal supports can be thought of as elements of a set T; then every temporal support corresponds to a finite subset $F = \{t_1, \ldots, t_n\}$, $t_1 < t_2 < \cdots < t_n$ of T. To every $t \in T$, a copy L_t of L is assigned. A history filter is a finite sequence $(a_{t_1}, \ldots, a_{t_n})$ of elements of L, corresponding to the temporal support $(t_1, \ldots, t_n)$ to points of T in which the filter is "active." We will map the history filter $(a_{t_1}, \ldots, a_{t_n})$ to the element $a_{t_1} \otimes \cdots \otimes a_{t_n}$ of the tensor product $L_{t_1} \otimes \cdots \otimes L_{t_n}$, where $L_{t_i} = L$ for every $t_i \in T$. The space of all histories then can be realized as the direct limit of the directed system of all finite tensor products, $(L_F, \{f_{FG}: L_F \to L_G, F \subseteq G, F, G$ finite subsets of $T\})$, where $L_F = L_{t_1} \otimes \cdots \otimes L_{t_n}$, $F = (t_1, \ldots, t_n)$. In order not to lose information, it is appropriate to consider a directed system $(L_F, \{f_{FG}: L_F \to L_G, F \subseteq G\})$ such that every f_{FG} is a closed morphism. It can be realized if we consider a D-poset L which has an ordering set of states, and state tensor products of copies of L.

Now let us return to the standard approach. Isham (n.d.) suggests the following strategy. Let $\alpha = (\alpha_{t_1}, \ldots, \alpha_{t_n})$ be a history filter with the support $(t_1, \ldots, t_n)$, where α_{t_i}, $i \leq n$, are projections in a Hilbert space H. Let us represent it with the product

$$\theta(\alpha_{t_1}, \ldots, \alpha_{t_n}) := \alpha_{t_1} \otimes \cdots \otimes \alpha_{t_n}$$

which acts on the tensor product space $\otimes_{t \in \{t_1,\ldots,t_n\}} H_t$ of n copies of H. The map $\theta: \oplus_{t \in \{t_1,\ldots,t_n\}} \mathcal{B}(H)_t \to \otimes_{t \in \{t_1,\ldots,t_n\}} \mathcal{B}(H)_t$ is many-to-one, but it becomes one-to-one when restricted to the subspace of projection operators. With the aid of the map θ, the operator representation $\Pi_{t \in \{t_1,\ldots,t_n\}} P(H)_t$ of the space of homogeneous histories $\mathcal{U}(t_1, \ldots, t_n)$ with temporal support $(t_1 \ldots, t_n)$ is embedded in the space $P(\otimes_{t \in \{t_1,\ldots,t_n\}} H_t)$ of projection operators on the Hilbert space $\otimes_{t \in \{t_1,\ldots,t_n\}} H_t$.

To incorporate arbitrary supports we need to collect together the operator algebras $\otimes_{t \in \{t_1,\ldots,t_n\}} \mathcal{B}(H)_t$ for all supports $(t_1, \ldots, t_n)$. It can be done by using an infinite tensor product of copies of $\mathcal{B}(H)$.

Let Ω denote a family of unit vectors in the Cartesian product $\Pi_{t \in T} H_t$ of copies of H labeled by the time values $t \in T$; i.e., Ω is a map from T to the unit sphere in H. The *infinite tensor product* $\otimes^{\Omega}_{t \in T} H_t$ based on Ω is defined to be the set of functions $\nu: T \to H$ such that $\nu(t) = \Omega(t)$ for all but a finite number of values t. The set of all such functions is given the usual pointwise vector space structure by defining $(ax + by)(t) = ax(t) + by(t)$, $a, b \in \mathbb{C}$, $x, y \in \otimes^{\Omega}_{t \in T} H_t$, and the scalar product

$$\langle x, y \rangle := \prod_{t \in T} \langle x(t), y(t) \rangle_H$$

where $\langle \cdot, \cdot \rangle_H$ is the inner product in the Hilbert space H. It is well defined because only a finite number of terms contribute to the product. It is a standard result that the resulting space is a Hilbert space (Guichardet, 1972).

An infinite tensor product $\otimes^{\Omega}_{t\in T} \mathcal{B}(H)_t$ is defined to be the weak closure of the set of all functions from T to $\mathcal{B}(H)$ that are equal to the unit operator for all but a finite set of t values.

Let $(\mathcal{T}, \leq)$ be the set of all temporal supports $(t_1, \ldots, t_n) \subset T$, $t_1 < t_2 < \cdots < t_n$, partially ordered by set inclusion. For $u \in \mathcal{T}$, let $\mathcal{P}(u)$ denote the set of all projections in $\otimes_{t\in v} \mathcal{B}(H)_t$. For $u, v \in \mathcal{T}$, $u \leq v$, let the mappings f_{uv}: $\otimes_{t\in u} \mathcal{B}(H)_t \to \otimes_{t\in v} \mathcal{B}(H)_t$ be the ampliations [i.e., $f_{uv}(\otimes_{t\in u} A_t)$ acts as A_t on the coordinates $t \in u$, and as the unit operators on coordinates $t \in v \backslash u$ on product vectors in $\otimes_{t\in v} H_t$]. The restriction of f_{uv} to $\mathcal{P}(u)$ maps $\mathcal{P}(u)$ to $\mathcal{P}(v)$, and f_{uv}: $\mathcal{P}(u) \to \mathcal{P}(v)$ is an injective homomorphism (of orthomodular lattices). Moreover, $(\mathcal{P}(u), f_{uv}$: $u, v \in \mathcal{T}$, $u \leq v)$ is a directed system. Let $(\mathcal{P}, f_u$: $u \in \mathcal{T})$ denote the direct limit.

For every $u \in \mathcal{T}$, $\otimes_{t\in u} P_t$ can be considered as a well-defined projection operator on the product space $\otimes^{\Omega}_{t\in u} H_t$. Let g_u: $\mathcal{P}(u) \to L(\otimes^{\Omega}_{t\in u} H_t)$, where $L(\otimes^{\Omega}_{t\in u} H_t)$ is the projection lattice in $\otimes^{\Omega}_{t\in u} H_t$. Clearly, $g_v \circ f_{uv} = g_u$. Hence $\mathcal{P}$ can be embedded into $L(\otimes^{\Omega}_{t\in u} H_t)$.

A modification of a decoherence functional may be obtained as follows. A decoherence functional is a function d: $\mathcal{U} \times \mathcal{U} \to \mathcal{C}$ satisfying the following conditions:

(a′) $d(0, \alpha) = 0$ for all α.
(b′) Hermicity: $d(\alpha, \beta) = \bar{d}(\beta, \alpha)$ for all α, β.
(c′) Positivity: $d(\alpha, \alpha) \geq 0$ for all α.
(d′) Additivity: if $\alpha \perp \beta$, then, for all γ, $d(\alpha \oplus \beta) = d(\alpha, \gamma) + d(\beta, \gamma)$.
(e′) Normalization: If $\alpha^1, \alpha^2, \ldots, \alpha^N$ and $\beta^1, \beta^2, \ldots, \beta^M$ are two complete sets of histories, then

$$\sum_{i=1}^{N} \sum_{j=1}^{M} d(\alpha^i, \beta^j) = 1$$

Then a decoherence functional behaves as a probability measure on any consistent complete set of histories containing no element orthogonal to itself.

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